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Strategic Bargaining and Competitive Bidding in a Dynamic Market Equilibrium

MELVYN G. COLES
CSIC, Barcelona and University of Essex

and

ABHINAY MUTHOO
University of Essex

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This paper extends the bargaining and matching literature, such as Rubinstein and Wolinsky (1985), by considering a new matching process. We assume that a central information agency exists, such as real estate agencies in the housing market and employment agencies (or newspapers) in the labour market, which puts traders into direct contact with each other. With heterogeneity of trader preferences, equilibrium trade is characterized by existing traders on each side of the market trying to match with the flow of new traders on the other side (since existing traders have already sampled and rejected each other). Two procedures of trade co-exist, namely a strategic bilateral bargaining process and a competitive bidding process, depending on the number of potential matches a new trader obtains. We characterize the unique symmetric Markov perfect equilibrium to this stochastic trading game.

1. INTRODUCTION

A central theme in the bargaining and matching literature has been to understand how the procedure of trade and the process by which traders meet affects the price outcome in a decentralized market equilibrium. Diamond (1981) and Mortensen (1982) were the first to characterize such a trading equilibrium, using the Nash bargaining solution to model the terms of trade. Rubinstein and Wolinsky (1985) extended this work by determining the terms of trade through an alternating offers bargaining procedure. Subsequent work has extended their analysis in several directions and, in particular, has focused on whether such trading processes can reproduce the competitive outcome.[1]

In this paper, we extend the above work by analysing bargaining in a market equilibrium with non-random search. For example, house-buyers do not search randomly when looking for a house to buy; they first visit an estate agent to obtain information on which houses are for sale.[2] Indeed, when a new buyer first visits the estate agency, there is typically a large *stock* of houses available for sale. If lucky, he might find his perfect home there, buy and immediately leave the market. However, if a suitable house is not presently for sale, he has to wait for a new house to come onto the market. Our essential point is

1. See Osborne and Rubinstein (1990) for a review of this literature.
2. Similarly, job seekers first check newspapers and employment agencies on job vacancies before searching for employment.

that with a central information agency, the matching process is characterized by *new* buyers immediately contacting the current *stock* of sellers. Those new buyers who do not find a match in the initial stock of sellers, subsequently contact the *flow* of later, *new* sellers. Similarly for sellers. We will show that in such a trading environment, the equilibrium price not only depends on the stocks of sellers and buyers currently in the market (as is commonly found), but also depends critically on the *rate* of flow of new buyers and sellers into the market. The presence of central information agencies dramatically affects the nature of the matching process, which not surprisingly, changes the nature of the equilibrium trade prices.

A natural feature of the above trading mechanism is that the negotiated price may now be determined *either* by a strategic bilateral bargaining procedure (if the new agent matches with only a single agent out of the present stock) *or* by a Bertrand-type game (if the new agent matches with more than one agent out of the present stock). This is similar in spirit to Wolinsky (1988), who also allows for "excess demand" at any point in time to be translated into instantaneous bidding competition.

Another important feature of our model is that the number of buyers and sellers in the market changes through time, according to a stationary Markov chain. As a result, because it is shown that the size of the market affects the agreed prices, the equilibrium trade prices also change through time. However, despite such complications, we are able to establish several equilibrium results, including: (i) "old" agents are strictly worse off than "new" ones; (ii) "old" agents are worse off as the stock of old agents increases; (iii) in the Bertrand-type game, the new agent negotiates a better price the more "old" agents there are on the other side of the market; (iv) with pairwise arrivals, old agents are better off as the arrival rate of new agents on the other side of the market increases,[3] but with independent arrivals this is not necessarily true; and (v) with pairwise arrivals, new agents are better off (on average) as the arrival rate of new agents increases.

These results are discussed in detail in the text. They largely result from the relative impacts of *instantaneous competition* (between old agents) and *intertemporal competition* (across new agents). By this latter term we mean that old agents always have the option to trade later, hoping that they will soon enter a bilateral bargaining relationship with a future new agent on the other side of the market. The greater the value of this outside option, the less each agent will bid in the Bertrand game. Of course, as the stock of old agents increases, or as the arrival rate of new agents decreases, the more distant this event becomes, and thus the old agents are in a comparably worse bargaining position.

The core theoretical property of this trading framework is that not all potential traders are in the market at the same time. If a trader cannot match with the traders currently present in the market, then that trader uses the option to stay in the market and trade with a later entrant. It is this option which implies that all traders (on average) are better off with higher turnover, as high turnover reduces the expected waiting time for two traders to meet each other and trade. We shall show that as turnover becomes arbitrarily high, the limiting price is one-half in all trades; that is, traders split the surplus equally. The intuition for this is that (with equal inflows), over any positive period of time, the number of buyers and sellers entering the market is arbitrarily large. Hence, regardless of the initial (finite) stocks of buyers and sellers, the expected proportion of buyers and sellers is (approximately) the same after an arbitrarily short period of time, and so, not surprisingly, the surplus is always shared equally.

3. In the Bertrand-type game, a new agent cannot negotiate such a good price as the reservation price of old agents moves against him.

In contrast, the standard matching literature also assumes that current traders face matching frictions to meet each other. Typically, constant returns to matching are assumed. It is this latter assumption which severs the link between prices and turnover rates. With constant returns to matching, an increase in the entry rate of new traders causes the stock of traders to increase, but, as each trader's contact rate with other traders is unchanged, this does not affect the market outcome. In particular, the equilibrium prices are independent of turnover.[4] In that literature, the focus is not on increasing turnover rates but on decreasing the discount rate.[5] It has been established that agents on the short side of the market negotiate a better price as the discount rate decreases, and furthermore, obtain more than half the surplus (cf., for example, Rubinstein and Wolinsky (1985) and Gale (1987)).

A natural interpretation of our model is that it introduces a matchmaker—the centralized information agency—into the matching framework. Although not modelled here, providers of such a service should be able to extract rents from the traders.[6] For example, real estate agents typically charge commissions and advertising in newspapers is costly. Related work on matchmakers include Bloch and Ryder (1994) and Yavas (1994). However, they study pure search models in which prices are *exogenously* specified. A different type of a central agency is the market maker who buys and sells the good that is being traded, and furthermore, sets ask and bid prices at which trade takes place. Brokers in the stock-market constitute an example, and clearly the pricing dynamics are quite different to those examined here (see Rubinstein and Wolinsky (1987), Moresi (1990), Gehrig (1993) and Wooders (1994), who analyse the potential role of such market makers in a matching framework).

Although the paper presented here is not linked directly to the auction literature, it offers some potentially useful insights into that literature. The auction literature typically considers one shot models, but clearly there is turnover in auction markets. For example, consider the buyer who wishes to obtain a Van Gogh sunflower painting, and suppose one is currently up for auction.[7] When bidding for this particular work, the potential buyer in the face of fierce bidding competition may drop out, realising that a similar piece may come up for auction at some future date when there may be less bidding competition. But the value of this outside option depends on the expected frequency with which such items come up for sale. If such works rarely appear on the market, the buyer faces a "now or never" decision—the value of this outside option is correspondingly small and his reservation price is correspondingly high. But, if such works arrive frequently onto the auction market, his reservation price is lower reflecting the value of that outside option. The fact that rare pieces create such excitement when they come up for auction suggests that there is an interesting dynamic dimension to such auctions.

The paper is organized as follows. In the next section we provide a description of the market model, which in effect is a stochastic dynamic game. The subsequent sections will then characterize a symmetric Markov perfect equilibrium for this market model, where Section 3 establishes the existence and uniqueness of such an equilibrium, and derives some basic characteristics of the equilibrium price outcome and dynamics. Comparative

4. See Pissarides (1990) for a review of this literature. Notice that this argument does not hold if there are increasing returns to matching. In that case, higher turnover leads to greater stocks and therefore faster contact rates—see, for example, Diamond (1982, 1984).

5. Or equivalently, increasing the matching rate. In the model presented in this paper with no matching frictions, reducing the discount rate is equivalent to increasing turnover.

6. However, in the U.K. labour market, for example, job centres provide their services for free, as such agencies are government funded.

7. Presumably, the original owner has had a preference shift—either needs the money or prefers to collect something else (or has died).

"statics" are obtained in Section 4, and the long run properties are derived in Section 5. The analysis in Sections 3–5 is based on the (analytically tractable) assumption that a new buyer and a new seller *jointly* arrive into the market according to some Poisson process. In Section 6, we explore (via numerical simulations) the effect on our results of the alternative assumption that new buyers and new sellers arrive *independently* of each other. Most of our proofs are relegated to the Appendix.

2. THE MODEL

The agents in the model are of two different types: sellers (s) and buyers (b). Each seller has one unit of an indivisible good for sale and each buyer wants to buy one unit of this good. A seller's valuation of her unit is zero. As the good under consideration is differentiated, a buyer assigns different values to different goods. Moreover, due to heterogeneity in the buyers' preferences, the buyers may differ from each other as to their respective valuations of a particular good. As is standard in the matching literature, the value a buyer places on a seller's good is considered as an independent random draw from some underlying (exogenous) distribution. For simplicity, we assume the distribution is binomial, where the valuation equals one with probability λ ($0<\lambda<1$), and zero otherwise.

The market operates over an infinite sequence of discrete points in time. At each point in time t, where $t=\ldots, -2\Delta, -\Delta, 0, \Delta, 2\Delta, \ldots$ with $\Delta>0$ (but small), the market contains a finite[8] stock N_t (where $N_t=0, 1, 2, \ldots$) of *old* agents of type s and an equal stock N_t of *old* agents of type b, where an agent is defined to be "old" if she was in the market at time $t-\Delta$. A single pair of new agents, one of each type, jointly arrive in the market at each point in time with some (small) probability $\alpha\Delta$, where $\alpha>0$ denotes the (pairwise) arrival rate.[9]

If at time t a pair of new agents do not arrive in the market, then nothing happens at time t (see the remark below), and the market evolves to time $t+\Delta$. But if a new pair do arrive, then immediately through a central information medium, which is not explicitly modelled here, the new agents are put into direct contact with the stock of old agents in the market. The new agent of each type then discovers whether there exists a gain to trade with any of the old agents of the opposite type.

If there do not exist any gains to trade between the new agents and the stock of old agents, then nothing further happens at time t and the market evolves to time $t+\Delta$.

However, if there exists a unit surplus between a new agent and exactly one old agent of the opposite type, then immediately the pair start bargaining over the partition of the unit surplus. We shall refer to this situation as the *bargaining* game. One of the players is chosen randomly (with equal probability) to propose a partition of the unit surplus. "Acceptance" terminates the game; the pair reach a deal and leave the market. On the

8. Assuming atomistic traders rather than a continuum of agents implies that the aggregate number of traders evolves stochastically over time. Although this complicates the model somewhat, it allows us to analyse how variations in the current number of traders, and how expectations on the future evolution of N, affect equilibrium trade prices. Furthermore, Section 6 demonstrates that the equilibrium trading dynamics trace out a so-called Beveridge curve. This is a particularly striking result as this curve is typically assumed to identify the so-called matching function.

9. Unfortunately, analytical tractability requires that we assume that a new seller and a new buyer always arrive together. An example of such an arrival process in the real estate market is the housemover who is both a buyer and a seller. In the labour market, if a worker quits his job or is fired, then the job and the worker may re-enter the labour market. Following divorce, both individuals may re-enter the marriage market. Section 6 relaxes this assumption but has to use numerical simulations to describe the reduced form properties of a market equilibrium.

other hand, "rejection" dissolves the match. In either case the market then evolves to time $t+\Delta$.

Furthermore, if there exists a unit surplus between a new agent and more than one old agent of the opposite type, then immediately this set of old agents engage in a *Bertrand-type* game: they simultaneously submit offers of partitions of a unit surplus and, subsequently, the new agent either rejects all such offers or she accepts exactly one offer. If two or more offers are identical and if the new agent wants to accept such an offer, then she accepts each such offer with equal probability. Acceptance of an offer means that a deal has been reached; the new agent and one old agent leave the market. Rejection of all offers dissolves the matches. In either case the market then evolves to time $t+\Delta$.

We assume that if both of the new agents can gainfully match with old agents, then the respective (Bertrand or bargaining) games are played *simultaneously*. In addition, we assume that agents involved in one game do not know whether another game is in fact being played.

The agents discount future payoffs, after they have entered the market, at a common rate $r>0$. Without loss of generality, we assume that if an agent reaches an agreement at time t, then her payoff from this deal is realised at the end of this "time period", at time $t+\Delta$. The payoff, therefore, to an agent from reaching an agreement at time t which gives her a share x of a unit surplus, discounted to time t, is $x/(1+r\Delta)$, where $x\in[0, 1]$. Finally, we assume that every aspect of the model is common knowledge amongst all agents. This completes the description of our model.

2.1. *A remark*

There is one important assumption that we have implicitly made which we shall now discuss. It will be shown below that, in any symmetric Markov perfect equilibrium, if a gain to trade exists between say a new seller and an old buyer, then the pair will trade immediately and leave the market. Only if no gain to trade exists does the new seller remain in the market, and of course becomes "old". But this implies that, in any such equilibrium, there never exists a gain to trade between the two sets of old agents.

However, we have implicitly assumed that if a gain to trade exists between a new seller and an old buyer *and* they do not trade immediately, then they cannot attempt to trade with each other in the future. This assumption is necessary for the model to remain tractable. Without such an assumption, we would have to describe some additional history-dependent trading procedures, where the dimension of the corresponding state space becomes very large. Proving the "immediate trade" property of the symmetric Markov perfect equilibria in such an extended model is complex. Although a different assumption would affect the exact characterization of equilibrium, we suspect it would not change the qualitative nature of the results obtained here.[10]

2.2. *The equilibrium concept*

The model that we have described is, in effect, a stochastic dynamic game. At each point in time, the "stage" game that is played, or the "stage" games that are played simultaneously, if one is played at all, and the set of players who play the game(s) are determined

10. Finally, note that we have also implicitly assumed that the pair of new agents cannot trade with each other. Again, this simplifies the model and the subsequent analysis. This assumption can, for example, be justified by assuming that there are no gains to trade between a pair of newly arrived agents (cf. the examples described in the previous footnote).

randomly. The probabilities associated with the set of feasible "events" at each time t are determined by the parameters Δ, α and λ, and by the state of the dynamic game at time t, namely the stock N_t of old agents of type i ($i=b, s$). For example, the probability that no game is played at time t is $(1-\alpha\Delta)+\alpha\Delta(1-\lambda)^{2N_t}$. In this paper we analyse the symmetric perfect equilibria in Markov pure strategies.

A pure strategy for an agent of either type is a complete plan of action, for each and every eventuality the agent may find herself after entering the market, and it can depend on the personal history of the agent in arbitrary and complicated ways. A Markov (i.e. payoff-relevant) pure strategy has the property that the action of an agent at any point in time t will depend on history only through its effect on the stock N_t of old agents of type i ($i=b, s$).

Formally, a Markov pure strategy σ for an agent of either type is defined by six functions $\langle P^n, R^n, P^o, R^o, T^n, T^o \rangle$ as follows:

(i) For each stock $N \geqq 1$ of old agents of either type, $P^k(N)$ denotes the share of a unit surplus offered by the agent when she is k (where $k=n$(new), o(old)), in the bargaining game, to an agent of the opposite type who is l (where $l \neq k$, $l=n$(new), o(old)). We emphasize that $P^k(N)$ is the share to the responder, while $1-P^k(N)$ is the share to the proposer. Moreover, $R^k(N, x)$ denotes the agent's response ("accept" or "reject") in such a bargaining game to an offer giving her a share x, where $x \in [0, 1]$.

(ii) For each stock $N \geqq 2$ of old agents of either type, $T^o(N)$ denotes the share of a unit surplus offered by the agent when she is old, in the Bertrand game, to the new agent of the opposite type. We emphasize that $T^o(N)$ denotes the share of the new agent, while $1-T^o(N)$ is the share of the old agent. Moreover, $T^n(N, x^m)$ denotes the response ("accept" or "reject") of the agent when she is new, in the Bertrand game, to the maximum x^m of the shares offered by the old agents of the opposite type, where $x^m \in [0, 1]$.

Given the symmetric nature of our model, we shall characterize those perfect equilibria in which all agents of both types employ the same Markov pure strategy. A symmetric Markov perfect equilibrium is a Markov pure strategy σ, employed by all agents of either type, such that for an arbitrary agent (of either type) the strategy σ is optimal (at any state $N \geqq 1$) given that all other agents of both types will play according to σ. For convenience we shall call a symmetric Markov perfect equilibrium a *Market Equilibrium* (ME, for short).

3. MARKET EQUILIBRIUM

Given a ME σ, let $V(N; r, \alpha, \Delta, \lambda)$ denote the value (expected payoff) to an old agent when the market contains $N \geqq 1$ old agents of each type, and let $W(N; r, \alpha, \Delta, \lambda)$ be the value to a new agent. Throughout the rest of this section, we simplify the notation by subsuming reference to the parameters r, α, Δ and λ. The following proposition much simplifies our analysis.

Proposition 1. *For any* $r>0$, $\alpha>0$, $\Delta>0$, $0<\lambda<1$ *and* ME σ: *for all* $N \geqq 1$, $R^o(N, P^n(N)) = R^n(N, P^o(N)) =$ *"accept", and for all* $N \geqq 2$, $T^n(N, T^o(N)) =$ *"accept".*

Proof. In the appendix. ||

Proposition 1 says that any ME is efficient: that is, if a gain to trade is possible between a new agent and (some) old agents, then, in equilibrium, the new agent will trade with probability one. The following results are an immediate consequence to Proposition 1 (and are formally derived in the appendix).

Corollary 1. *For any* $r>0$, $\alpha>0$, $\Delta>0$, $0<\lambda<1$ *and* ME σ:

(a) *for all* $N\geqq 1$, *in the bargaining game the new and old agent obtain the same expected surplus* 0·5, *and*
(b) *for all* $N\geqq 2$, $T^o(N)=1-(1-\lambda)^N V(N)-[1-(1-\lambda)^N]V(N-1)$.

Corollary 1(a) occurs because, in the bargaining game, both the new agent and the old agent have the same equilibrium "disagreement" payoff and each is assumed to make the price offer with probability one half. Part 1(b) occurs since each old agent bids her equilibrium reservation value in a Bertrand-type game. With probability $1-(1-\lambda)^N$, the other new agent does match (and there are $N-1$ old agents next period), while with probability $(1-\lambda)^N$, the other new agent does not match (and there are N old agents next period).

Corollary 2. *For any* $r>0$, $\alpha>0$, $\Delta>0$, $0<\lambda<1$ *and* ME σ, $V(N)$ *must satisfy the difference equation*

$$\left(\frac{r}{\alpha}\right)V(N)=\lambda(1-\lambda)^{N-1}[\tfrac{1}{2}-V(N)]+(1-\lambda)^{2N}[V(N+1)-V(N)]$$
$$+[1-(1-\lambda)^{N-1}][1-(1-\lambda)^N][V(N-1)-V(N)], \tag{1}$$

subject to the initial condition

$$\left(\frac{r}{\alpha}\right)V(1)=\lambda[\tfrac{1}{2}-V(1)]+(1-\lambda)^2[V(2)-V(1)]. \tag{2}$$

Corollary 2 can be derived using standard dynamic programming methods, using Corollary 1 for the appropriate trading prices. The flow value to being an old agent (given N) depends on three terms. The first is that the old agent is the only one to match with the new agent (and hence gets expected surplus 0·5 in the bargaining game), the second is that no matches occur (and hence N increases to $N+1$) and the third is that both new agents match and that the new agent opposite our old agent can also match with another old agent (so that there is no bilateral bargaining and N decreases to $N-1$). The initial condition (2) can be obtained by putting $N=1$ in (1).

Corollary 3. *For any* $r>0$, $\alpha>0$, $\Delta>0$, $0<\lambda<1$ *and* ME σ, $V(N)$ *must satisfy the following inequalities*:

$$(1-\lambda)^N V(N+1)+[1-(1-\lambda)^N]V(N)\leqq\tfrac{1}{2}(\forall N\geqq 1), \tag{3}$$
$$(1-\lambda)^N V(N+1)+V(N)+[1-(1-\lambda)^N]V(N-1)\leqq 1(\forall N\geqq 2), \tag{4}$$
$$0\leqq V(N)\leqq 1(\forall N\geqq 1). \tag{5}$$

Inequalities (3) and (4), respectively, are necessary conditions for trade to occur in the bargaining game and the Bertrand game, while (5) simply says that an old agent can guarantee a payoff of zero by never trading, and can obtain at most the whole unit surplus instantaneously.

Identifying a ME has effectively been reduced to showing that there exists a solution to the difference equation (1) which satisfies the initial condition (2) and the trading requirements (3), (4) and (5).

Theorem 1. *For any* $r>0$, $\alpha>0$, $\Delta>0$ *and* $0<\lambda<1$ *there exists a unique* ME σ: *for any* $N\geqq 1$ *and* $x\in[0, 1]$, $R^o(N, x)=R^n(N, x)=R(N, x)$ *and* $P^o(N)=P^n(N)=P(N)$, *where*

$$R(N, x)=\begin{cases}\text{"accept"} & \text{if } x\geqq(1-\lambda)^N V(N+1)+[1-(1-\lambda)^N]V(N),\\ \text{"reject"} & \text{if } x<(1-\lambda)^N V(N+1)+[1-(1-\lambda)^N]V(N),\end{cases}$$

and $P(N)=(1-\lambda)^N V(N+1)+[1-(1-\lambda)^N]V(N)$; *for any* $N\geqq 2$ *and* $x^m\in[0, 1]$,

$$T^n(N, x^m)=\begin{cases}\text{"accept"} & \text{if } x^m\geqq(1-\lambda)^N V(N+1)+[1-(1-\lambda)^N]V(N),\\ \text{"reject"} & \text{if } x^m<(1-\lambda)^N V(N+1)+[1-(1-\lambda)^N]V(N),\end{cases}$$

and $T^o(N)$ *is defined in Corollary* 1(b), *where* $V(\cdot)$ *is the unique solution to* (1)–(5).

Proof. In the appendix. ||

Proposition 2. *For any* $r>0$, $\alpha>0$, $\Delta>0$ *and* $0<\lambda<1$, *in the unique* ME, $V(N)\in(0, 1/2)$ *for all* $N\geqq 1$.

Proof. In the appendix. ||

A central aspect of the ME is the (associated) equilibrium value function $V(\cdot)$ for old agents. This value function is particularly interesting since, by Corollary 1(b), it also approximates the equilibrium trading price in the Bertrand game (assuming $V(N+1)\approx V(N)$). As this value function encapsulates much that is of economic interest, we focus our discussion of the ME by investigating the properties of $V(\cdot)$, where we now take into account that $V(\cdot)=V(N, r, \alpha, \lambda)$ defined by (1)–(5).

Most of the propositions which follow can be understood by considering the interplay between *instantaneous* competition among the old agents at any point in time (when a new agent arrives) and *intertemporal* competition (across new agents). By this latter term, we mean that old agents always have the option to trade later, hoping that they will soon enter a (bilateral) bargaining game with a future new agent (where they will then get expected surplus 0·5). Clearly, the greater the value of this outside option, the less each old agent will bid in the Bertrand game. This will depend on N and the underlying parameters r, α and λ. For example, one might expect that *smaller* values of r increase the effects of intertemporal competition (by reducing discounting).

Proposition 3. *For any* $r>0$, $\alpha>0$, $\Delta>0$ *and* $0<\lambda<1$, *in the unique* ME, $V(N)$ *is strictly decreasing in* N, *and* $V(N)\to 0$ *as* $N\to\infty$.

Proof. In the appendix. ||

As N increases, the expected payoff to an *old* agent decreases. This result is consistent with the notion that greater competition should translate into a relatively lower payoff to the agent concerned. There are two basic reasons for this result. First, given an old agent can match with a new agent, an increase in N makes it more likely that this old agent is competing in the Bertrand game rather than playing the bargaining game. This makes the old agent worse off. Moreover, the matching dynamics are such that if N is high today, then it is expected to remain high in the future.[11] Hence, the value of the old agent's outside option (to wait to trade in the future) also decreases as N increases. In the Bertrand game, old agents will bid even more competitively. These reinforcing effects result in old agents being worse off as N increases.

It is interesting to note that, for any finite $N \geqq 2$, the outcome of the Bertrand game does not give an old agent a zero payoff as, perhaps, one would *a priori* expect, but gives her a positive surplus (since $V(N) > 0$ implies that $T^o(N)$, defined in Corollary 1(b), is strictly less than one). This, of course, is due to the intertemporal competition effect. An old agent expects with strictly positive probability to be in a bargaining game with some new agent in the future, where she would then obtain a positive surplus (equal to one-half). However, in the limit as $N \to \infty$, this possibility becomes a *distant* event: hence $V(N) \to 0$ (and $T^o(N) \to 1$) as $N \to \infty$.

Let $W(N)$ denote the equilibrium expected payoff (value) of being a new agent in this market when the market contains N old agents of each type. Proposition 4 now states some properties of this function. In particular, it compares it to $V^*(N)$ defined as the equilibrium expected payoff to an old agent *conditional* on the arrival of a new pair of agents.

Proposition 4. *For any $r > 0$, $\alpha > 0$, $\Delta > 0$ and $0 < \lambda < 1$, in the unique* ME, $W(1) = V^*(1)$ *and for all $N \geqq 2$, $W(N) > V^*(N)$. Furthermore, $W(N) \to 1$ as $N \to \infty$ and $\Delta \to 0$.*

Proof. It is straightforward to compute these functions in terms of $V(N)$. The Proposition then follows directly from Propositions 2 and 3. ||

When $N = 1$, the new and old agent are in the same bargaining position and so $W(1) = V^*(1)$. However, if $N \geqq 2$, then the new agent can potentially match with other old agents which makes her strictly better off. The essential difference is that old agents have previously sampled the market but failed to match with anyone, while a new agent has many old agents with whom she could potentially strike a deal. Of course, this advantage is lost should she fail to match with anyone.

Having found a basic characterization of $V(N)$ and $W(N)$, given r, α and λ, we now turn to see how varying these parameters affect the value functions of both types of agents.

4. COMPARATIVE STATICS

By inspection of (1)–(5), it follows that the equilibrium value functions $V(\cdot)$ and $W(\cdot)$ depend on r and α only through r/α. Thus we consider comparative statics on the equilibrium functions $V(N, r/\alpha, \lambda)$ and $W(N, r/\alpha, \lambda)$.

Proposition 5. *For any $r > 0$, $\alpha > 0$, $\Delta > 0$ and $0 < \lambda < 1$, in the unique* ME, $V(N, r/\alpha, \cdot)$ *is strictly decreasing in r/α for all $N \geqq 1$.*

Proof. In the appendix. ||

This result arises because of two separate, but reinforcing effects. Suppose, for example, that the arrival rate of new agents increases. By reducing the expected time before

11. These dynamics will be discussed in detail in the next section.

possible trade occurs, this increases the expected value of being an old agent. Moreover, it also improves the value of each old agent's outside option in the Bertrand game and hence their equilibrium payoff increases in that game. By increasing intertemporal competition across new agents, an old agent's expected payoff increases.

Although the effect of r/α on $V(\cdot)$ is unambiguous, it is perhaps surprising that its effect on $W(\cdot)$ is ambiguous. One might have thought that if old agents are better off (say with an increase in α), then new agents, who are on the other side of the market, must be worse off. This is clearly true if the new agent matches with more than one old agent, where their terms of trade are determined in the Bertrand game. However there is a second countervailing effect. It is also possible that the new trader does not match at all, and subsequently becomes an old agent. An increase in α makes the new trader better off in this state. Which effect dominates depends on N. If N is small (for example, if $N=0$), then the latter effect dominates. If, on the other hand, N is arbitrarily large, the former effect dominates.

Proposition 6. *For any $\Delta>0$, $0<\lambda<1$ and $N\geqq 1$, in the unique* ME, $V(N)\rightarrow 1/2$ *(resp., $V(N)\rightarrow 0$) as $r/\alpha\rightarrow 0$ (resp., $r/\alpha\rightarrow\infty$).*

Proof. The monotonicity of $V(N, r/\alpha, \cdot)$ in r/α for all $N\geqq 1$ combined with the boundedness of $V(N)$ for all $N\geqq 1$ implies that, as $r/\alpha\rightarrow 0$ (resp., as $r/\alpha\rightarrow\infty$), for each $N\geqq 1$, $V(N)$ must converge to its supremum (resp., to its infimum), denoted by $\bar{V}(N)$ (resp., $\underline{V}(N)$). Clearly, for all $N\geqq 1$, $0\leqq \underline{V}(N)<\bar{V}(N)\leqq 1/2$. Using (1) and (2) it follows that, for all $N\geqq 1$, $\underline{V}(N)=0$ and $\bar{V}(N)=1/2$. ‖

This result is quite intuitive. As $r/\alpha\rightarrow 0$, the intertemporal competition effect neutralizes the instantaneous (Bertrand) competition effect: a new and an old agent have equal "bargaining power". Consequently, in the Bertrand game, the new and an old agent split the unit surplus equally (note that $T^o(N)\rightarrow 1/2$ as $r/\alpha\rightarrow 0$). Indeed, by computing $W(N, r/\alpha, \cdot)$, it follows that $W(N, r/\alpha, \cdot)\rightarrow 1/2$ as $r/\alpha\rightarrow 0$ and $\Delta\rightarrow 0$ for all $N\geqq 1$. In contrast, as $r/\alpha\rightarrow\infty$, the intertemporal effect is non-existent; in the Bertrand game, the new agent has all the "bargaining power", and obtains the whole surplus ($T^o(N)\rightarrow 1$).

We now turn to the effects of λ (how easy it is to match) on the equilibrium payoffs. In this case, the total effect turns out to be ambiguous. There are two basic effects on $V(N)$. The first is that an increase in λ increases the arrival rate of acceptable matches, $\alpha\lambda$. This effect makes an old agent better off. However, given N, an increase in λ also implies that if an acceptable match occurs for an old agent, that agent is more likely to be in the Bertrand game than the bargaining game, which makes the agent worse off.

The effect on $W(N)$ is also ambiguous. An increase in λ makes it easier to match (given N) which makes the new agent better off. But if the new agent fails to match, the agent becomes an old agent and hence may be worse off.

5. THE LONG RUN

Having characterized the equilibrium payoffs and actions, we now consider the equilibrium market dynamics. The immediate trade property and preference structure imply the following equilibrium transition probabilities.

Proposition 7. *Conditional on a new trading pair, the equilibrium transition probabilities for $N\geqq 1$ are $P(N\rightarrow N+1)=(1-\lambda)^{2N}$, $P(N\rightarrow N)=2(1-\lambda)^N[1-(1-\lambda)^N]$ and $P(N\rightarrow N-1)=[1-(1-\lambda)^N]^2$, while $P(0\rightarrow 1)=1$.*

To understand these dynamics, a useful concept is the matching function $M(N, \lambda)$, defined as the expected number of matches conditional on a new trading pair entering the market. Proposition 7 implies that $M(N, \lambda)=2[1-(1-\lambda)^N]$, which is strictly increasing in N—a new agent is more likely to match if there is a greater number of old agents. Moreover, due to congestion effects typically discussed in the matching literature, this function exhibits diminishing returns to N. There is also a unique value N^c which satisfies $M(N^c, \lambda)=1$, which can be rearranged as $(1-\lambda)^{N^c}=1/2$. We shall refer to N^c as the ergodic centre. For $N>N^c$, the expected number of matches conditional on a new pair entering the market is greater than one. In that case, the number of old agents of each type is expected to decrease. Similarly, if $N<N^c$, then the expected number of matches (conditional on a new pair of traders) is less than one, and thus, the number of old agents of each type is expected to increase. The equilibrium dynamics are stable, reverting in expected value towards the ergodic centre.

Let $\langle \pi_N \rangle_{N=0}^{\infty}$ denote the ergodic distribution of N, where π_N is the probability that at some particular date in the infinite future there are N old agents of each type in the market. Standard probability theory implies

$$(1-\lambda)^{2N}\pi_N=[1-(1-\lambda)^{N+1}]^2\pi_{N+1}$$

for $N \geqq 0$.[12] Notice that the ergodic distribution depends only on λ. Direct inspection shows that the ergodic distribution must be unimodal, where, if N^* denotes the mode, then $N^*-1<N^c<N^*+1$. That is, the mode is within one of the ergodic centre.[13]

It also follows that the ergodic distribution with small λ first order stochastically dominates one with a larger λ. This occurs because for any $N>0$ and conditional on the arrival of a new pair of agents, N is more likely to decrease with higher values of λ. Hence N is more likely to take low values.[14]

TABLE 1

Numerical simulations of the ergodic distribution

λ	mode	mean	st. dev.	$[N_L, N_H]$
0·5	1	1·1	0·7	[0, 2]
0·13	5	5·1	1·4	[3, 8]
0·027	25	25·1	3·0	[19, 31]
0·013	53	53·1	4·3	[45, 62]

To obtain some insight into this distribution, we briefly consider some numerical examples. Table 1 describes the mode, mean, standard deviation and symmetric 95% confidence interval $[N_L, N_H]$ where $P(N<N_L)=P(N>N_H)=0{\cdot}025$. We consider values of $\lambda=0{\cdot}5$, 0·13, 0·027 and 0·013 (which imply that $N^c=1$, 5, 25 and 53, respectively).

As λ decreases (matching is less likely), the expected number of traders in the market increases and its value is always close to the ergodic centre. A less obvious result is that as λ decreases, the ergodic variance increases. The intuition for this is best understood at the ergodic centre. Assume $\lambda \leqq 1/2$ so that $N^c \geqq 1$, and for simplicity assume N^c takes an

12. See Feller (1968, chapter 15.7). This condition corresponds to $\pi_N P(N \to N+1)=\pi_{N+1}P(N+1 \to N)$. This distribution can be constructed by first obtaining the sequence $\langle \hat{\pi}_N \rangle$ where $\hat{\pi}_0=1$ and $\hat{\pi}_N$ satisfies the above difference equation. If $S=\sum_{N=0}^{\infty} \hat{\pi}_N$, then $\pi_N=\hat{\pi}_N/S$.

13. The conditions $\pi_{N^*-1}<\pi_{N^*}$ and $\pi_{N^*}>\pi_{N^*+1}$, and the above difference equation imply that N^* must satisfy $(1-\lambda)/(2-\lambda)<(1-\lambda)^{N^*}<1/(2-\lambda)$. From these inequalities, it follows that $(1-\lambda)^{N^*-1}>1/2>(1-\lambda)^{N^*+1}$, which is the required condition.

14. For any given $N>0$, the ratio π_{N+1}/π_N is strictly decreasing in λ; i.e. for a higher value of λ, π_N decreases more rapidly with N. This implies first order stochastic dominance.

integer value. If $N=N^c$, then a new trader matches with probability $1/2$, while if $N=N^c+1$, then a new trader matches with probability $1/2+\lambda/2$. The extra term is the probability that the new trader cannot match with the original N^c traders, but can match with the extra one. For $N=N^c+1$, λ large implies N is more likely to fall straight back to N^c (when the next trading pair arrive). By increasing the strength of the mean reversion, a high λ results in a low ergodic variance. The two extreme cases make this point transparent. $\lambda=1$ implies $\pi_0=\pi_1=1/2$ and the market switches between $N=0$ and $N=1$. Conversely, as $\lambda\to 0$, the dynamics limit to a random walk, where an extra old trader in the market hardly affects the matching probability of a new trader.

The effect of α on the distribution of N is quite different. Suppose that, at $t=0$, $N=N_0$. Over time, the distribution of N_t converges to the ergodic distribution. An increase in α implies faster convergence.

We briefly consider a numerical simulation which demonstrates how these effects interact. Assume that Δ is arbitrarily small so that the entry of new trading pairs is described by a Poisson process. Set λ equal to 0·13 and consider α equal to either 365, 60 or 10. Assuming an annual discount rate of $r=0{\cdot}06$, these arrival rates correspond to a pair of new traders arriving (on average) every day, every 6 days and every 36 days. With $\lambda=0{\cdot}13$, the arrival rate of a match (for an old trader) is once a week, once every 6 weeks and once every 9 months. Figure 1 describes $V(N, r/\alpha)$ for each of these values.[15] It also graphs $\hat{V}(N, r/\alpha)$ defined by

$$(r/\alpha)\hat{V}=\lambda(1-\lambda)^{N-1}[1/2-\hat{V}],$$

which implies $\hat{V}(N, r/\alpha)=(\lambda/2)(1-\lambda)^{N-1}/[r/\alpha+\lambda(1-\lambda)^{N-1}]$.

Clearly reflecting Propositions 2 and 3, V is increasing in α and decreasing in N. For $\alpha=365$, V is very close to one-half, reflecting Proposition 6. Although an arrival rate of

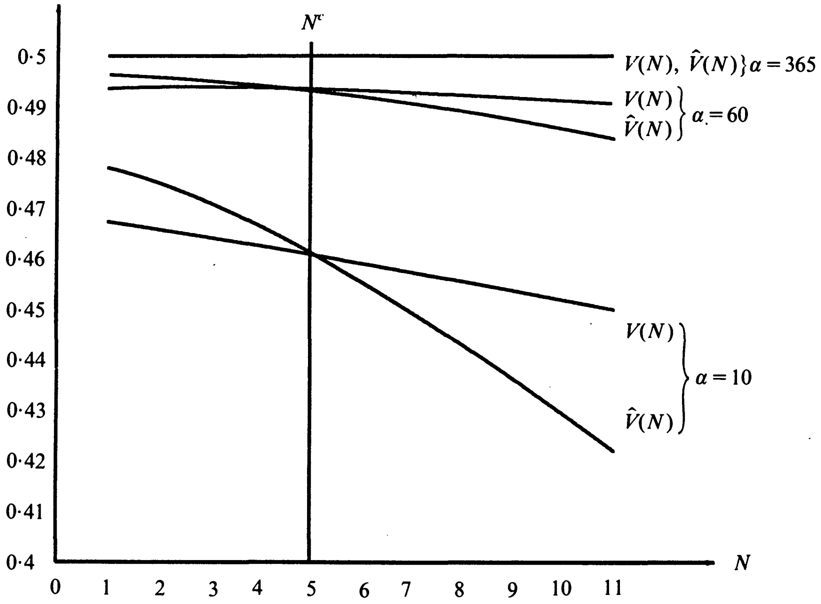

FIGURE 1

15. We only draw these graphs for $0<N<11$ as [3, 8] is the symmetric 95% confidence interval for N.

$\alpha = 10$ is quite small, the discount rate $r = 0{\cdot}06$ implies that $r/\alpha = 0{\cdot}006$ remains small. Thus V remains relatively close to one-half even for reasonably low turnover numbers.

Notice that $\hat{V}$ solves (1) assuming no N dynamics—it is a "myopic" solution which depends only on the current rate at which an old trader expects to enter the bilateral bargaining game. V is different as agents expect N to change over time.

As in all other simulations that have been run but not reported here,[16] $V > \hat{V}$ for $N \geqq N^c + 1$. For such N, traders expect their numbers will fall over time and so the rate at which each trader will enter the bilateral bargaining game is expected to rise over time. This raises V above $\hat{V}$. The converse is the case for $N \leqq N^c - 1$. At the ergodic centre $V(N^c, \cdot) \approx \hat{V}(N^c, \cdot) = [1 + 2r(1-\lambda)/(\alpha\lambda)]/2$, which reflects the fact that there is no systematic bias on the expected dynamics.[17] This decomposition implies that α has two main effects on V. The first is that, for given N, it instantaneously increases the rate at which an old trader enters the bilateral bargaining game. This is reflected by the shift in $\hat{V}$. Secondly, given N, it increases the rate at which the distribution of N converges to the ergodic distribution. This is reflected by the difference in V and $\hat{V}$. This latter effect is greater the further N is away from N^c.

Given the above approximation for V at the ergodic centre, then for r/α small (the empirically relevant case), V can be further approximated as[18]

$$V(N^c, r/\alpha) \approx 1/2 - (r/\alpha\lambda)(1-\lambda).$$

This approximation shows how the Bertrand trading price depends directly on expected turnover. At the ergodic centre, the difference between the bilateral bargaining price (one-half) and the Bertrand price doubles if the turnover rate halves. Price risk is much greater for old agents in low turnover markets where Bertrand prices are sensitive to changes in expected turnover rates. This result is quite different to a standard matching model. In such a model, if the arrival rate of traders doubles, then, in a steady state with constant returns to matching, the number of traders also doubles. As this leaves the contact rate unaffected, the equilibrium trading prices do not depend on turnover.

All of the above discussion has focused on old traders. It is straightforward to calculate the expected value of being a new trader. For the above parameter values, the value of being a new trader increases with (the three values of) α for $N < 9$, but decreases with α for $N \geqq 9$. The intuition is straightforward. If a new trader cannot match with the current set of old traders, then the new trader becomes an old trader and, in that state, is better off with an increase in α. Conversely, if the new trader can match with more than one old trader, the new trader is better off with a low α as this drives down the old agents' reservation value. With exactly one match, the expected payoff is always $1/2$. Hence, given N, the overall effect of α on the value of being a new trader depends on the relative likelihood that there are no matches rather than many. The smaller the value of N, the more likely it is that the new trader does not have a match and so prefers a high α. But N evolves endogenously. Over the symmetric 95% confidence interval in the above

16. All simulations used GAUSS and the programmes are available on request from the authors.

17. In all simulations performed, $V(N)$ falls (almost) linearly across the ergodic center, so that $V(N^c - 1) - V(N^c) \approx V(N^c) - V(N^c + 1)$. As $(1-\lambda)^{N^c} = 1/2$, (1) implies $V(N^c) \approx \hat{V}(N^c)$. If we had assumed a continuum of agents, so that there are no aggregate dynamics, then $\hat{V}(N^c)$ identifies the equilibrium payoff at the steady state and reduces to $V^{**} = [1 + 2r/\alpha^{**}]/2$ (where with a continuum of agents, we would have to assume Poisson rather than binomial preferences (otherwise there is a zero measure of old traders in the steady state) and $\alpha^{**} \equiv \alpha\lambda$ is the arrival rate of matches, when $\alpha\Delta$ is the mass of new traders who enter the market each period and Δ is arbitrarily small.)

18. Except for markets with very low turnover, simulations find that this is a good approximation, typically being accurate to 3 significant figures.

simulations, both old and new traders are better off with high turnover rates. Values of N sufficiently large to make Bertrand competition more likely also imply that N is likely to fall when a new trading pair enters the market. Thus, the reduced form matching process ensures that such N occur with (relatively) low probability. Indeed it can be argued that at the ergodic centre, new agents are always better off with an increase in α. Assuming $N=N^c$, new agents fail to match with probability one-half (and prefer a high α to raise $V(N)$) while with probability strictly less than one-half[19] they have more than one match (and prefer a low α to lower $V(N)$). The greater probability weight on "no matches" implies they prefer a high α.

6. AN EXTENSION: INDEPENDENT ARRIVALS

The analysis in the preceding sections has assumed that a new buyer always arrived with a new seller. The implied symmetry allowed analytical results. This section considers the more general and perhaps more realistic case that new buyers and new sellers arrive independently. Unfortunately, this extension is not analytically tractable and we have to rely on using numerical simulations to obtain insights into a ME. Although some results are different, they continue to be driven by the same intuition: prices are determined by an old agent's outside option to wait to trade with a future new agent in a bilateral monopoly situation.

Assume at each point in time a single new agent arrives with probability $\alpha\Delta$ (Δ small), where $\alpha>0$ is the arrival rate of agents. Given a new agent has arrived, the new agent is either a buyer or a seller with equal probability. Let B_t and S_t denote the stocks of old buyers and old sellers present in the market at time t. We again characterize the symmetric Markov perfect equilibria (ME, for short) for this game. We shall omit a formal definition of a Markov pure strategy since it is similar to that in the basic model (cf. Section 2.2). A Markov pure strategy is one in which the action of an agent at any point in time depends on history only through its effect on the current stock B of old buyers and the stock S of old sellers.

For any ME and state (B, S), let $V(B, S)$ denote the value (expected payoff) to an *old buyer*. By symmetry, the value to an *old seller* equals $V(S, B)$. Using similar arguments to those in the proof to Proposition 1, it follows that any ME is efficient.[20] Given that, it is now straightforward to obtain the analogous equation to (1) for $V(B, S)$.

Fix a ME and a state (B, S). Conditional on a new seller entering the market (at an arbitrary time t), the expected surplus (realised at the end of the "time period", at time $t+\Delta$) received by an old buyer in the bargaining game equals $[1-V(S+1, B)+V(B, S+1)]/2$. Furthermore, the expected payoff to an old buyer in the Bertrand game equals $V(B-1, S)$. Using standard dynamic programming arguments it follows that $V(\cdot)$ satisfies

$$\begin{aligned}2\left(\frac{r+\alpha}{\alpha}\right)V(B, S)&=(1-\lambda)^S V(B+1, S)+[1-(1-\lambda)^S]V(B, S-1)\\&\quad+\frac{\lambda(1-\lambda)^{B-1}}{2}[1-V(S+1, B)+V(B, S+1)]\\&\quad+[1-(1-\lambda)^{B-1}]V(B-1, S)+(1-\lambda)^B V(B, S+1), \qquad (6)\end{aligned}$$

19. It can be shown that this probability must be less than 0·35 at the ergodic centre.

20. That is, if a gain to trade is possible between a new agent and (some) old agents, then in equilibrium the new agent will trade with probability one.

subject to the trading inequalities

$$V(S+1, B) + V(B, S+1) \leqq 1, \tag{7}$$

$$V(S+1, B) + V(B-1, S) \leqq 1, \tag{8}$$

$$0 \leqq V(B, S) \leqq 1. \tag{9}$$

Identifying a ME requires showing there exists a solution to the above system of equations. Unfortunately, the asymmetry of the bargaining game now implies that $V(B, S)$ depends on $V(S+1, B)$—the threatpoint of a seller in the bargaining game. (6) defines a two dimensional partial difference equation which is of unbounded order (where $B, S \geqq 0$). There is no obvious analytical approach here. However, it is possible to solve this system numerically.[21]

To obtain some insight on the numerical simulations described below, consider the equilibrium (B, S) dynamics. The independent arrivals assumption implies that $B-S$ follows a random walk where given (B_0, S_0) at $t=0$, $E_0[B_t - S_t] = B_0 - S_0$. Hence, the expected value of (B_t, S_t) lies along a 45° line passing through (B_0, S_0). Now consider $B_t + S_t$. Conditional on a new arrival at time t, it follows that

$$E[B_{t+\Delta} + S_{t+\Delta}] = B_t + S_t + 0{\cdot}5[(1-\lambda)^{B_t} - [1-(1-\lambda)^{B_t}]] + 0{\cdot}5[(1-\lambda)^{S_t} - [1-(1-\lambda)^{S_t}]].$$

Consider the middle term. With probability one half, the new agent is a seller. With probability $(1-\lambda)^{B_t}$ this new seller does not match and the stock of agents increases by one, while with probability $1-(1-\lambda)^{B_t}$ the new seller matches and the stock of agents decreases by one. Similarly if a new buyer enters. Rearranging implies

$$E[B_{t+\Delta} + S_{t+\Delta}] = B_t + S_t + (1-\lambda)^{B_t} + (1-\lambda)^{S_t} - 1.$$

As before, we define the "ergodic centre" where $E[B_{t+\Delta} + S_{t+\Delta}] = B_t + S_t$. Dropping the t subscript, the above implies that the ergodic centre is the curve $(1-\lambda)^B + (1-\lambda)^S = 1$. This relationship can be rewritten as $S = F(B)$, where $F(\cdot)$ is a strictly positive, strictly decreasing and strictly convex function. If (B, S) lie above this ergodic centre, then $B+S$ is expected to fall over time and so the expected value of (B_t, S_t) declines along the 45° line to the ergodic centre. Conversely, if $B+S$ is below the ergodic centre, $B+S$ is expected to increase over time, and so the expected value of (B_t, S_t) increases along the 45° line. In a probabilistic sense, (B_t, S_t) will stay close to this curve through time. Furthermore, the fact that $B_t - S_t$ follows a random walk implies that the observed sequence $\{B_t, S_t\}$ will tend to wander up and down this curve, and hence these variables will covary negatively. In the labour market literature, the observed negative covariance between the stock of unemployed (sellers) and the stock of vacancies (buyers) is known as the Beveridge curve. Typically, this relationship is explained by assuming matching frictions in the market. The above implies that such a statistical relationship may exist even when there are no such trading frictions. The ergodic centre defines a Beveridge curve.

21. Simulations revealed that this difference equation exhibits unstable roots as $B, S \to \infty$. Since $V(\cdot)$ must be bounded, a numerical solution was found by choosing a grid of size $(\bar{B}+1) \times (\bar{S}+1)$ where these numbers were large relative to the symmetric ergodic centre. (6) was then solved using the method of successive over-relaxation (an extension of Gauss-Seidal; cf., for example, Gerald and Wheatly (1984)) using the boundary condition $V(\bar{B}+1, S) - V(\bar{B}, S) = V(\bar{B}, S) - V(\bar{B}-1, S)$ (and similarly for the $S=\bar{S}$ boundary) which successfully ruled out the explosive roots. Convergence was assumed when the last sweep over the grid resulted in no value changing by more than 10^{-6}. At convergence, iterations at the ergodic centre were of the order 10^{-8}.

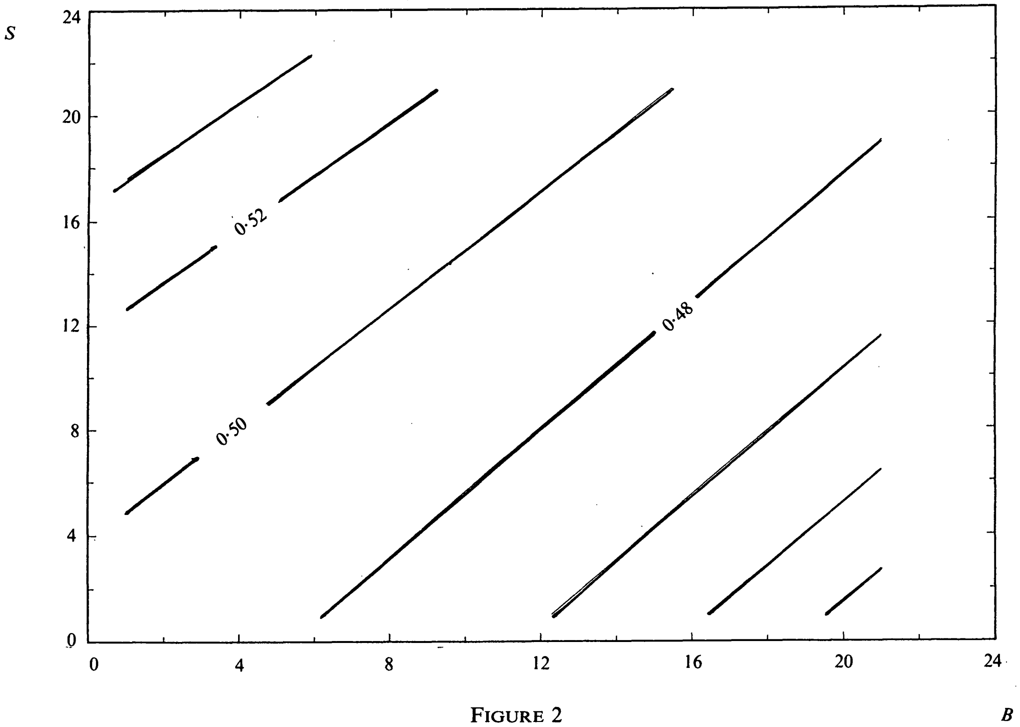

FIGURE 2

Contour plot $V(B, S) = V_0 : r = 0{\cdot}06,\ \alpha = 120,\ \lambda = 0{\cdot}13$

With this understanding of the expected matching dynamics, we now turn to our simulations. Many simulations were run but we shall discuss only one (as its results are typical). Figure 2 describes the contour plot $V(\cdot)$ for parameter values $r = 0{\cdot}06$, $\alpha = 120$ and $\lambda = 0{\cdot}13$ (which are the same parameter values as in the previous simulation, where before we assumed agents arrived in pairs and $\alpha = 60$).[22] The choice of λ implies that the ergodic centre passes through the point (5, 5).

The first thing to notice is that $V(\cdot)$ is increasing in S and decreasing in B. This happened in all of our simulations. The intuition is that conditional on a new seller entering the market, an increase in B reduces the probability that an old buyer will enter a (bilateral) bargaining game, which makes each old buyer worse off. Conversely, an increase in S reduces the probability that a new buyer fails to match, and so the stock of old buyers B_t will tend to fall over time. This raises the expected value of being an old buyer.

A second point is that it is no longer true that $V(\cdot) < 1/2$ necessarily. This occurs because of the asymmetry of the agents' "threatpoints" in the bargaining game. In particular, if B is small and S is large, the difference in the threatpoints, where $V(B, S) > V(S, B)$, implies that the old buyer negotiates a greater share of the surplus in the bargaining game. Assuming $V(\cdot)$ is always increasing in S and decreasing in B, then the $\lim_{S\to\infty} V(1, S)$ provides us with an upper bound for $V(B, S)$. Assuming such a limit exists, (6) implies that $\lim_{S\to\infty} V(1, S) = 1/[1 + (4r/\alpha\lambda)]$.[23] Notice that this upper bound is close to one for small values of r/α.

22. The numerical grid was 101×101, but we focus here around the symmetric ergodic centre which for $\lambda = 0{\cdot}13$ is $B = S = 5$.

23. It can also be shown that $\lim_{B\to\infty} V(B, S) = 0$.

A third point is that the contours are steeper than 45°. This feature also occurred in all of our simulations. To understand this, suppose at $t=0$, (B_0, S_0) lies above the ergodic centre. The above has established that as t increases $(E_0(B_t), E_0(S_t))$ falls along the 45° line to the ergodic centre. That is, B is expected to decrease over time and so the rate at which each old buyer enters the bilateral bargaining game is expected to increase gradually. Hence points (B_0, S_0) along the 45° line with lower values of B_0 imply old buyers not only have a higher rate of entry into the bilateral bargaining game today, but also in the entire future. This makes them better off. Simulations also show that these contours become flatter and tend to 45° as α becomes large. The faster the equilibrium dynamics are expected to converge to the ergodic centre, the smaller the loss through B being high today.

Finally, we consider the effect of decreasing r/α (e.g. by increasing the arrival rate of new traders). Previously Proposition 5 implied that $V(\cdot)$ increased unambiguously. This is no longer true. Assuming a limit exists, it is easy to show analytically that for any finite state of the market (B, S), $V(B, S)\to 0$ (resp., 1/2) as $r/\alpha\to\infty$ (resp., 0). Simulations confirmed this result.[24] But the example given in Figure 2 shows that $V(B, S)>1/2$ for some (B, S) and r/α. Unlike in the pairwise arrivals model, V is not monotonic in r/α. The reason for this is the asymmetry of the bargaining game in the independent arrivals model. A decrease in α can make an old buyer better off if this weakens the bargaining position of a new seller sufficiently in the bargaining game. For example, if we use the same parameter values as above except now $\alpha=30$, then simulation shows that $V(1, 2)$ increases from 0·505 to 0·520. The threatpoint of the new seller in the bargaining game $V(13, 1)$ decreases from 0·46 to 0·35, which is clearly a large effect. This non-monotonicity was found to occur only for small B and large S, where the asymmetry in the bargaining game is most in favour of old buyers.

APPENDIX

Proposition 1.

In order to prove this proposition we first establish several preliminary results, which are organized into seven lemmas. Furthermore, in order to simplify and shorten our argument, we shall make use (in the proofs to Lemmas A.1 and A.2) of the following "tie-breaking" assumption.

Assumption 1. Consider an arbitrary agent who has to choose between a pair of actions a_1 and a_2 at some time t. Suppose that action a_1 gives the agent a payoff of u_1 at time $t+\Delta$, while action a_2 will give her a payoff of u_2 at time $t+2\Delta$. If $u_1=u_2/(1+r\Delta)$, then the agent (although indifferent between these two actions) will prefer to choose action a_1.

Lemma A.1. *Fix $r>0$, $\alpha>0$, $\Delta>0$ and $0<\lambda<1$. Let an arbitrary* ME *σ and an arbitrary stock $N\geqq 1$ be given. $R^o(N, P^n(N))=$"accept" if and only if $R^n(N, P^o(N))=$"accept". Furthermore, if $R^o(N, P^n(N))=$"accept", then $P^n(N)=P^o(N)$.*

Proof. Fix a ME and a stock $N\geqq 1$, and consider the bargaining game. Suppose that $R^o(N, P^n(N))=$ "accept". This implies that $P^n(N)\geqq d(N)$ and that $1-P^n(N)\geqq d(N)$, where $d(N)$ denotes the equilibrium payoff to both the old agent and the new agent if either of these agents rejects any offer. By adding these inequalities, it follows that $d(N)\leqq 1/2$. If $d(N)<1/2$, then it is trivial to verify (by appealing to perfection) that $R^n(N, P^o(N))=$ "accept". To establish the same if $d(N)=1/2$, we appeal to Assumption 1. A symmetric argument establishes the converse: $R^n(N, P^o(N))=$"accept" implies that $R^o(N, P^n(N))=$"accept". We now establish the second statement in the lemma. First recall that $R^o(N, P^n(N))=$"accept" implies that $d(N)\leqq 1/2$. If $d(N)<1/2$, then it is

24. Notice that $\lim_{(r/\alpha)\to 0}\lim_{S\to\infty} V(1, S)=1$, which differs from $\lim_{S\to\infty}\lim_{(r/\alpha)\to 0} V(1, S)=1/2$. The limiting value of $V(1, S)$ depends on the order in which the limits are taken.

trivial to verify (by appealing to perfection) that $P^n(N)=P^o(N)=d(N)$. To establish the same if $d(N)=1/2$, we appeal to Assumption 1. ‖

Lemma A.2. *Fix $r>0$, $\alpha>0$, $\Delta>0$ and $0<\lambda<1$. Let an arbitrary* ME σ *and an arbitrary stock $N\geqq 2$ be given. If $R^n(N, P^o(N))=$"accept", then $T^n(N, T^o(N))=$"accept".*

Proof. Fix a ME and a stock $N\geqq 2$, and consider the Bertrand-type game. Our argument is by contradiction. Thus, suppose $T^n(N, T^o(N))=$"reject". This implies that $T^o(N)\leqq d(N)$, where $d(N)$ denotes the equilibrium payoff to an old agent and to the new agent following the new agent's rejection of all offers in the Bertrand-type game. Clearly, $d(N)$ is also the equilibrium "disagreement" payoff in the bargaining game. Hence, (as we have shown above in the proof to Lemma A.1) $R^n(N, P^o(N))=$"accept" implies that $d(N)\leqq 1/2$. If $d(N)<1/2$, then, in the Bertrand game, an old agent can make a profitable, unilateral, "one-shot deviation" by instead offering the new agent a share x of a unit surplus such that $d(N)<x<1-d(N)$. To establish the same (that is, the existence of a profitable, unilateral, "one-shot deviation") if $d(N)=1/2$, we appeal to Assumption 1. ‖

Lemma A.3. *Fix $r>0$, $\alpha>0$, $\Delta>0$ and $0<\lambda<1$. Let an arbitrary* ME σ *and an arbitrary stock $N\geqq 2$ be given. If $R^o(N, P^n(N))=$"accept" and $T^n(N, T^o(N))=$"accept", then:*

$$T^o(N)=1-(1-\lambda)^N V(N)-[1-(1-\lambda)^N]V(N-1), \tag{A.3.1}$$

and

$$(1-\lambda)^N V(N+1)+V(N)+[1-(1-\lambda)^N]V(N-1)\leqq 1. \tag{A.3.2}$$

Proof. Fix a ME and a stock $N\geqq 2$, and consider the Bertrand game between a new agent and $m\geqq 2$ old agents of the opposite type. In equilibrium, with probability $1/m$ an arbitrary old agent leaves the market with the new agent, but with probability $(m-1)/m$ she is not chosen by the new agent and hence remains in the market. In the former case her payoff (at the end of the "time period") equals $1-T^o(N)$, while in the latter case her expected payoff (at the end of the same "time period") will equal $(1-\lambda)^N V(N)+[1-(1-\lambda)^N]V(N-1)$, since with probability $(1-\lambda)^N$ there does not exist any surplus between the *other* new agent and any one of the old agents of the opposite type, while with probability $1-(1-\lambda)^N$ there does and in equilibrium (in both the bargaining and Bertrand games) trade occurs. It is easy to verify that "Bertrand-type competition" implies that these two payoffs must be equal, and hence $T^o(N)$ is as stated in Lemma A.3. Moreover, in equilibrium, $T^o(N)\geqq(1-\lambda)^N V(N+1)+[1-(1-\lambda)^N]V(N)$, for otherwise the new agent would reject the offer $T^o(N)$. The inequality stated in Lemma A.3 therefore follows after substituting for $T^o(N)$. ‖

Lemma A.4. *Fix $r>0$, $\alpha>0$, $\Delta>0$ and $0<\lambda<1$. Let an arbitrary* ME σ *and an arbitrary stock $N\geqq 1$ be given. If $R^o(N, P^n(N))=$"accept" and $T^n(N, T^o(N))=$"accept", then:*

$$\left(\frac{r}{\alpha}\right)V(N)=\lambda(1-\lambda)^{N-1}[\tfrac{1}{2}-V(N)]+(1-\lambda)^{2N}[V(N+1)-V(N)]$$
$$+[1-(1-\lambda)^{N-1}][1-(1-\lambda)^N][V(N-1)-V(N)], \tag{A.4.1}$$

where we define $V(0)=0$,

$$(1-\lambda)^N V(N+1)+[1-(1-\lambda)^N]V(N)\leqq \tfrac{1}{2}, \tag{A.4.2}$$

and (if $N\geqq 2$)

$$(1-\lambda)^N V(N+1)+V(N)+[1-(1-\lambda)^N]V(N-1)\leqq 1. \tag{A.4.3}$$

Proof. In order to compute $V(N)$, let us consider an arbitrary old agent (of either type) and suppose that a pair of new agents (of opposite types) have arrived. It is helpful to decompose the possible outcomes into three states:

(i) The new agent (of the opposite type to the old agent under consideration) can *only* match with the old agent under consideration. It follows from Lemma A.1 that the expected payoff to the old agent (realised at the end of the "time period") in this state is $1/2$. The probability that this state occurs is $\lambda(1-\lambda)^{N-1}$.

(ii) (Applies if $N\geqq 2$.) The new agent can match with the old agent under consideration and, also, with some other old agent. It follows from Lemma A.3 that the expected payoff to the old agent (realised at the end of the "time period") in this state is $(1-\lambda)^N V(N)+[1-(1-\lambda)^N]V(N-1)$. The probability that this state occurs is $\lambda[1-(1-\lambda)^{N-1}]$.

(iii) The old agent under consideration cannot match. In this case her expected payoff (realised at the end of the "time period") is $(1-\lambda)^{2N-1}V(N+1)+[1-(1-\lambda)^{N-1}]\ [1-(1-\lambda)^{N}]\ V(N-1)+[(1-\lambda)^{N-1}+(1-\lambda^{N})-2(1-\lambda)^{2N-1}]V(N)$, where we define $V(0)=0$. The probability that this state occurs is $(1-\lambda)$.

Let $V^*(N)$ denote the expected payoff to the old agent (realised at the end of the "time period") given that a new pair have arrived. Hence, $V(N)=[(1-\alpha\Delta)V(N)+\alpha\Delta V^*(N)]/(1+r\Delta)$. After using (i), (ii) and (iii) to substitute out $V^*(N)$, and then rearranging, it will follow that $V(N)$ is indeed as stated in equation (A.4.1).

Inequality A.4.3 is a restatement of the inequality stated in Lemma A.3, while Inequality A.4.2 states that the equilibrium disagreement payoff in the bargaining game to each agent must be less than or equal to one-half. ||

Lemma A.5. *Fix* $r>0$, $\alpha>0$, $\Delta>0$ *and* $0<\lambda<1$. *Let an arbitrary* ME σ *and an arbitrary stock* $N\geqq 1$ *be given. If* $R^o(N, P^n(N))=$ *"reject" and* $T^n(N, T^o(N))=$ *"reject", then*:

$$\left(\frac{r}{\alpha}\right)V(N)=V(N+1)-V(N), \tag{A.5.1}$$

and

$$V(N+1)>\tfrac{1}{2}. \tag{A.5.2}$$

Proof. Under the given (trade) hypotheses, the computation of $V(N)$ is trivial, since when a new pair of agents do arrive no agreement is reached whatsoever. $V(N)=[(1-\alpha\Delta)V(N)+\alpha\Delta V(N+1)]/(1+r\Delta)$. Hence, after rearranging, it follows that $V(N)$ is indeed as stated in equation (A.5.1). Inequality (A.5.2) describes a necessary condition for no trade in both the bargaining and the Bertrand games. ||

Lemma A.6. *Fix* $r>0$, $\alpha>0$, $\Delta>0$ *and* $0<\lambda<1$. *Let an arbitrary* ME σ *and an arbitrary stock* $N\geqq 2$ *be given. If* $R^o(N, P^n(N))=$ *"reject" and* $T^n(N, T^o(N))=$ *"accept", then*:

$$T^o(N)=1-[(1-\lambda)^N+N\lambda(1-\lambda)^{N-1}]V(N)-[1-(1-\lambda)^N-N\lambda(1-\lambda)^{N-1}]V(N-1), \tag{A.6.1}$$

$$[(1-\lambda)^N+N\lambda(1-\lambda)^{N-1}]V(N+1)+[1-(1-\lambda)^N-N\lambda(1-\lambda)^{N-1}]V(N)>\tfrac{1}{2}, \tag{A.6.2}$$

and

$$[(1-\lambda)^N+N\lambda(1-\lambda)^{N-1}]V(N+1)+V(N)+[1-(1-\lambda)^N-N\lambda(1-\lambda)^{N-1}]V(N-1)\leqq 1. \tag{A.6.3}$$

Proof. Similar to the proof of Lemma A.3. The difference is that, with probability $(m-1)/m$, when the old agent is not chosen by the new agent in the Bertrand game, the old agent's expected payoff is slightly more complex, since the *other* new agent will not reach a deal in the bargaining game, which occurs with probability $N\lambda(1-\lambda)^{N-1}$. Moreover, note that Inequality (A.6.2) is a necessary condition for no trade in the bargaining game, where the L.H.S. denotes the equilibrium disagreement payoff to each agent in the bargaining game. Inequality (A.6.3) is analogous to Inequality (A.3.2) and is a necessary condition for trade in the Bertrand game. ||

Lemma A.7. *Fix* $r>0$, $\alpha>0$, $\Delta>0$ *and* $0<\lambda<1$. *Let an arbitrary* ME σ *and an arbitrary stock* $N\geqq 1$ *be given. If* $R^o(N, P^n(N))=$ *"reject" and* $T^n(N, T^o(N))=$ *"accept", then*:

$$\left(1+\frac{r}{\alpha}\right)V(N)=[(1-\lambda)^N+N\lambda(1-\lambda)^{N-1}]^2[V(N+1)-V(N)]$$
$$+[(1-\lambda)^N+N\lambda(1-\lambda)^{N-1}]V(N)+[1-(1-\lambda)^N-N\lambda(1-\lambda)^{N-1}]^2V(N-1), \tag{A.7.1}$$

where we define $V(0)=0$,

$$[(1-\lambda)^N+N\lambda(1-\lambda)^{N-1}]V(N+1)+[1-(1-\lambda)^N-N\lambda(1-\lambda)^{N-1}]V(N)>\tfrac{1}{2}, \tag{A.7.2}$$

and (if $N\geqq 2$)

$$[(1-\lambda)^N+N\lambda(1-\lambda)^{N-1}]V(N+1)+V(N)+[1-(1-\lambda)^N-N\lambda(1-\lambda)^{N-1}]V(N-1)\leqq 1. \tag{A.7.3}$$

Proof. Similar to the proof of Lemma A.4. The differences being: (a) that in state (i) (see the proof of Lemma A.4), in the bargaining game, the expected payoff to the old agent is *not* 1/2 (since no trade occurs), but it is $[(1-\lambda)^N+N\lambda(1-\lambda)^{N-1}]V(N+1)+[1-(1-\lambda)^N-N\lambda(1-\lambda)^{N-1}]V(N)$; (b) that in state (ii), in the

Bertrand game, the expected payoff to the old agent is $1-T^o(N)$ where $T^o(N)$ is as stated in equation A.6.1; and (c) that in state (iii), when the old agent cannot match, her expected payoff equals $[(1-\lambda)^{N-1}+(N-1)\lambda(1-\lambda)^{N-2}][(1-\lambda)^N+N\lambda(1-\lambda)^{N-1}]V(N+1)+[1-(1-\lambda)^{N-1}-(N-1)\lambda(1-\lambda)^{N-2}][1-(1-\lambda)^N-N\lambda(1-\lambda)^{N-1}]V(N-1)+[(1-\lambda)^{N-1}+(N-1)\lambda(1-\lambda)^{N-2}+(1-\lambda)^N+N\lambda(1-\lambda)^{N-1}]V(N)$, where we define $V(0)=0$. Inequalities (A.7.2) and (A.7.3) are re-statements of inequalities (A.6.2) and (A.6.3). ||

Proof of Proposition 1.

Let an arbitrary ME σ be given. Lemmas A.1 and A.2 imply that the state space $\{1, 2, 3, \ldots\}$ can be partitioned into three mutually exclusive and exhaustive sets Ω_1, Ω_2 and Ω_3 such that (i) for any $N\in\Omega_1$, $R^o(N, P^n(N))=$ "reject" *and* $T^n(N, T^o(N))=$ "accept", (ii) for any $N\in\Omega_2$, $R^o(N, P^n(N))=$ "reject" *and* $T^n(N, T^o(N))=$ "reject", and (iii) for any $N\in\Omega_3$, $R^o(N, P^n(N))=$ "accept" *and* $T^n(N, T^o(N))=$ "accept".

Steps 1 and 2 below, respectively, establish that the sets Ω_1 and Ω_2 must both be finite. Hence Ω_3 must be infinite. In particular there will exist a finite N^* such that for any $N>N^*$, $N\in\Omega_3$. Step 3 then establishes, via a backward induction argument, that for *all* $N\geqq 1$, $N\in\Omega_3$. ||

Step 1. Ω_1 *is a finite set.*

Proof of Step 1. We argue by contradiction. Thus suppose Ω_1 is an infinite set. Hence, for any N_1^* there will exist an $N>N_1^*$ s.t. $N\in\Omega_1$. Thus, for such an N, (A.7.1)–(A.7.3) must hold. Since, for *any* N, $0\leqq V(N)\leqq 1$, by choosing N_1^* sufficiently large there will exist sufficiently small numbers ε_1, ε_2 and ε_3 such that (A.7.1)–(A.7.3) become:

$$\left(1+\frac{r}{\alpha}\right)V(N)=V(N-1)+\varepsilon_1 \tag{A.7.1a}$$

$$\varepsilon_2+V(N)>\tfrac{1}{2} \tag{A.7.2a}$$

$$\varepsilon_3+V(N)+V(N-1)\leqq 1. \tag{A.7.3a}$$

We emphasize that one can make the numbers ε_1, ε_2 and ε_3 *arbitrarily* small by choosing N_1^* sufficiently large. Now substituting for $V(N-1)$ in (A.7.3a) using (A.7.1a) we obtain that: $(\varepsilon_3-\varepsilon_1)+(2+r/\alpha)V(N)\leqq 1$. That is, $V(N)\leqq(1-\varepsilon_3+\varepsilon_1)/(2+r/\alpha)$. But this contradicts (A.7.2a) for N_1^* sufficiently large (i.e. for ε_1, ε_2 and ε_3 sufficiently small) since $r/\alpha>0$ would then imply that: $(1-\varepsilon_3+\varepsilon_1)/(2+r/\alpha)<1/2-\varepsilon_2$. ||

Step 2. Ω_2 *is a finite set.*

Proof of Step 2. We argue by contradiction. Thus suppose Ω_2 is an infinite set. Step 1 has established that Ω_1 must be a finite set. Ω_3 could be a finite or an infinite set. We divide our argument into these two cases.

Case 1: Ω_3 *is a finite set.*

In this case, therefore, there will exist an N_2^* such that for any $N>N_2^*$, $N\in\Omega_2$. Hence, for such N, (A.5.1) and (A.5.2) must hold. Using (A.5.1), we obtain that for all $K=1, 2, \ldots$: $V(N_2^*+K+1)=(1+r/\alpha)^K V(N_2^*+1)$. Clearly there will exist a K^*, sufficiently large, such that for any $K>K^*$: $V(N_2^*+K+1)<1/2$ (since $0\leqq V(N_2^*+1)\leqq 1$). But this contradicts (A.5.2).

Case 2: Ω_3 *is also an infinite set.*

Thus, for any N_3^* there will exist infinitely many $N>N_3^*$ such that $N\in\Omega_3$. For such N, (A.4.1) and (A.4.3) must hold. By a similar argument to that given in the proof of Step 1 above, we can establish that by choosing N_3^* sufficiently large there exists (arbitrarily) small numbers ε_1 and ε_2 such that for any $N>N_3^*$ and $N\in\Omega_3$, $V(N)\leqq(1-\varepsilon_2+\varepsilon_1)/(2+r/\alpha)$. Now $r/\alpha>0$ implies that $1/2>1/(2+r/\alpha)$. Hence, for N_3^* sufficiently large (i.e. for sufficiently small ε_1 and ε_2): $1/2>(1-\varepsilon_2+\varepsilon_1)/(2+r/\alpha)$. Consequently, there exists N_3^* sufficiently large such that for any $N>N_3^*$ and $N\in\Omega_3$, $V(N)<1/2$. However, for any $N>N_3^*$ and $N\in\Omega_2$, $V(N+1)>1/2$ (c.f. (A.5.2)). Let $N'>N_3^*$ be s.t. $N'\in\Omega_2$; such an N' must exist since Ω_2 is an infinite set. But then $V(N'+1)>1/2$, which implies that $(N'+1)\in\Omega_2$. In fact, by induction, it follows that for any $N\geqq N'$, $N\in\Omega_2$. This contradicts the hypothesis that Ω_3 is an infinite set. ||

Step 3. *For all* $N \geqq 1$, $N \in \Omega_3$.

Proof of Step 3. Steps 1 and 2 imply that there exists an $N^*(\geqq 1)$ such that for any $N > N^*$, $N \in \Omega_3$. We now employ a backward induction argument to establish Step 3: we *assume* that an *arbitrary* N (where $N = N^*+1, N^*, N^*-1, N^*-2, \ldots, 2$) is an element of the set Ω_3, and we shall *deduce* that $(N-1) \in \Omega_3$. Step 3 then follows by backward induction. We argue by contradiction. Thus suppose $(N-1) \notin \Omega_3$.

Case 1: $(N-1) \in \Omega_2$.

Hence, (A.5.1) applies: $(1+r/\alpha)V(N-1) = V(N)$. This implies that $V(N-1) - V(N) = -(r/\alpha)V(N-1)$, which implies that $V(N) \geqq V(N-1)$. Moreover, (A.5.2) applies: $V(N) > 1/2$. Now, since by assumption $N \in \Omega_3$ and $N \geqq 2$, (A.4.1) and (A.4.2) apply. Consider (A.4.1). $V(N) > 1/2$ implies that the L.H.S. > 0. Moreover, $V(N) > 1/2$ and $V(N) \geqq V(N-1)$ imply that the first term in the R.H.S. < 0 and the third term in the R.H.S. $\leqq 0$. Consequently, the second term in the R.H.S. > 0, and hence, $V(N+1) < V(N)$. Thus, $V(N) > 1/2$ implies that $V(N+1) > 1/2$. But this contradicts (A.4.2). Hence, our supposition that $(N-1) \in \Omega_2$ cannot be true.

Case 2: $(N-1) \in \Omega_1$.

In this case (A.7.1)–(A.7.3) apply. If $N-1=1$, then (A.7.1) and (A.7.2) become: $(1+r/\alpha)V(1) = V(2)$ and $V(2) > 1/2$. Hence, the same argument as in Case 1 goes through and establishes a contradiction. Consequently, $1 \notin \Omega_1$. Now suppose $N-1 \geqq 2$. After rearranging (A.7.1) we obtain:

$$\left(\frac{r}{\alpha}\right)V(N-1) = a^2[V(N) - V(N-1)] + (1-a)[(1-a)V(N-2) - V(N-1)], \qquad \text{(A.7.1ā)}$$

where $a = (1-\lambda)^N + N\lambda(1-\lambda)^{N-1}$.

First suppose $V(N) \leqq 1/2$. This implies, using (A.7.2), that $V(N-1) > 1/2$. Thus, using (A.7.1ā) above, it follows that $(1-a)V(N-2) > V(N-1)$. Consequently, $(1-a)V(N-2) + V(N-1) + aV(N) > 1$, since $V(N-1) > 1/2$. But this contradicts (A.7.3). *Now* suppose $V(N) > 1/2$. This implies, using (A.4.2), that $V(N+1) < 1/2$. Consequently, using (A.4.1), it follows that $V(N-1) > V(N)$. Hence, using (A.7.1ā) above, it follows that $(1-a)V(N-2) > V(N-1)$, leading, as before, to a contradiction. In conclusion, $(N-1) \notin \Omega_1$. ||

Proof of Theorem 1.

Theorem 1 follows immediately after we establish that there exists a unique solution (value function) $V: \{1, 2, 3, \ldots\} \to [0, 1]$ that satisfies (1)–(5). The strategy of our proof is as follows. From Corollary 2 we know that $V(N)$ must satisfy

$$\left(\frac{r}{\alpha}\right)V(N) = \lambda(1-\lambda)^{N-1}[\tfrac{1}{2} - V(N)] + (1-\lambda)^{2N}[V(N+1) - V(N)]$$
$$+[1-(1-\lambda)^{N-1}][1-(1-\lambda)^N][V(N-1) - V(N)], \qquad \text{(B.1)}$$

subject to the initial condition

$$\left(\frac{r}{\alpha}\right)V(1) = \lambda[\tfrac{1}{2} - V(1)] + (1-\lambda)^2[V(2) - V(1)]. \qquad \text{(B.2)}$$

Corollary 3 also requires that

$$V(N) \text{ is bounded for all } N. \qquad \text{(B.3)}$$

Lemma B.1 below first establishes that if a solution to (B.1), (B.2) and (B.3) exists, then it must be unique. Lemma B.2 then *constructs* such a solution. Finally, Lemma B.3 establishes that such a solution satisfies $0 < V(N) < 1/2$ for all $N \geqq 1$, and hence by inspection, it must also satisfy (3)–(5).

Lemma B.1. *If a solution to* (B.1)–(B.3) *exists, then it must be unique.*

Proof. We argue by contradiction. Thus suppose there exists at least two solutions to (B.1)–(B.3), and let us consider an arbitrary pair of such solutions, denoted by $\langle V_1(N)\rangle_{N=1}^{\infty}$ and $\langle V_2(N)\rangle_{N=1}^{\infty}$. Clearly, $V_1(1) \neq V_2(1)$, for otherwise, it follows from (B.1) and (B.2) that $V_1(N) = V_2(N)$ for *all* $N \geqq 1$.

We shall show (below) that for any T there exists an N^* s.t. for all $N>N^*$, $|V_1(N)-V_2(N)|>T$. This is a contradiction, since, by (B.3), there exists a finite T^* s.t. for *all* $N\geqq 1$, $|V_1(N)-V_2(N)|\leqq T^*$.

Define a function $G\colon \mathfrak{R}\to\mathfrak{R}^\infty$ as follows. For each $x\in\mathfrak{R}$, $G(x)$ is an infinite sequence of real numbers $\langle G_N(x)\rangle_{N=1}^\infty$, where $G_1(x)=x$ and the remainder of the sequence $\langle G_N(x)\rangle_{N=2}^\infty$ is defined recursively by:

$$\left(\frac{r}{a}\right)G_N(x)=\lambda(1-\lambda)^{N-1}[\tfrac{1}{2}-G_N(x)]+(1-\lambda)^{2N}[G_{N+1}(x)-G_N(x)]$$
$$+[1-(1-\lambda)^{N-1}][1-(1-\lambda)^N][G_{N-1}(x)-G_N(x)], \tag{B.1a}$$

subject to the initial condition

$$\left(\frac{r}{a}\right)x=\lambda[\tfrac{1}{2}-x]+(1-\lambda)^2[G_2(x)-x]. \tag{B.2a}$$

Notice that, by construction, the sequence $\langle G_N(x)\rangle$ is identical to the sequence $\langle V_i(N)\rangle$ (where $i=1,2$) iff $x=V_i(1)$. Step 1 below establishes that for each $x\in\mathfrak{R}$ and for $N\geqq 2$

$$\frac{dG_N(x)}{dx}>\left(\frac{r}{a}\right)(1-\lambda)^{-2N}.$$

Since in this expression the R.H.S. diverges to plus infinity as $N\to\infty$, it follows that for any $x_1, x_2\in\mathfrak{R}$ s.t. $x_1\neq x_2$ and for any T there exists an N^* such that for $N>N^*$,

$$|G_N(x_1)-G_N(x_2)|>T.$$

The desired result now follows by taking $x_1=V_1(1)$ and $x_2=V_2(1)$. ||

Step 1. *For each $x\in\mathfrak{R}$ and for each $N\geqq 2$:*

$$\text{(i)}\ \frac{dG_N(x)}{dx}>\left(\frac{r}{a}\right)(1-\lambda)^{-2(N-1)},\quad \textit{and}\quad \text{(ii)}\ \frac{dG_N(x)}{dx}>\frac{dG_{N-1}(x)}{dx}>1.$$

Proof of Step 1. The argument is by induction. First, using (B.2a), it follows that $dG_2(x)/dx=[r/a+\lambda+(1-\lambda)^2]/(1-\lambda)^2>(r/a)(1-\lambda)^2$; and moreover, note that $dG_2(x)/dx>1=dG_1(x)/dx$. Now *assume* that Step 1 is true for $N=2, 3, \ldots, k$, and we shall *deduce* that it is true for $k+1$.

From (B.1a) it is evident that $G_{k+1}(\cdot)$ depends on x through its independence on G_k and G_{k-1}. Hence, for each $x\in\mathfrak{R}$:

$$\frac{dG_{k+1}(x)}{dx}=\frac{\partial G_{k+1}}{\partial G_k}\frac{dG_k(x)}{dx}+\frac{\partial G_{k+1}}{\partial G_{k-1}}\frac{dG_{k-1}(x)}{dx}.$$

After obtaining $\partial G_{k+1}/\partial G_k$ and $\partial G_{k+1}/\partial G_{k-1}$ using (B.1a), and rearranging, we have:

$$\frac{dG_{k+1}(x)}{dx}=\left(1+(1-\lambda)^{-2k}\left[\frac{r}{a}+\lambda(1-\lambda)^{k-1}\right]\right)\frac{dG_k(x)}{dx}$$
$$+(1-\lambda)^{-2k}[1-(1-\lambda)^{k-1}][1-(1-\lambda)^k]\left[\frac{dG_k(x)}{dx}-\frac{dG_{k-1}(x)}{dx}\right].$$

By the inductive hypothesis, $dG_k(x)/dx>dG_{k-1}(x)/dx$ and $dG_k(x)/dx>1$. Hence, it follows that

$$\text{(i)}\ \frac{dG_{k+1}(x)}{dx}>\left(\frac{r}{a}\right)(1-\lambda)^{-2k},\quad\text{and}\quad \text{(ii)}\ \frac{dG_{k+1}(x)}{dx}>\frac{dG_k(x)}{dx}>1.\quad ||$$

Lemma B.2. *There exists a solution to* (B.1)-(B.3).

Proof. To construct a solution, we use the standard method of guessing a particular integral and then analysing the characteristic function.[25] We guess the particular integral

$$V_N=\textstyle\sum_{n=1}^\infty a_n(1-\lambda)^{nN}. \tag{B.4}$$

25. As $V(N)$ in (B.1) is described by a linear second order difference equation, it can be identified by solving for its particular integral and its complementary function. In particular, we will show that the characteristic function has the saddle path property, and so (B.3) implies we must choose the saddle path solution. We guess the particular integral and solve for it using the method of undetermined coefficients—see, for example, Sargent (1987) for such techniques.

Using the method of undetermined coefficients, it follows that

$$a_1=\frac{0{\cdot}5}{r(1-\lambda)/\alpha\lambda-1},$$

$$a_2=\frac{a_1\lambda(3-2\lambda)}{1-(1-\lambda)^2(1+r/\alpha)},$$

and, for $n>2$, a_n is defined by

$$a_n[1-(1+r/\alpha)(1-\lambda)^n]=a_{n-1}[2-\lambda-2(1-\lambda)^n]-a_{n-2}[1-\lambda-(2-\lambda)(1-\lambda)^{n-1}].$$

Since for $N\geqq 1$, $(1-\lambda)^{nN}$ declines exponentially as $n\to\infty$, the sum defined by (B.4) converges as long as a_n does not grow too quickly with n. Now as $n\to\infty$, the difference equation defining a_n converges to

$$a_n-(2-\lambda)a_{n-1}+(1-\lambda)a_{n-2}=0,$$

which has non-explosive roots $(1-\lambda)$ and 1. Hence for $0\leqq(1-\lambda)<1$, although a_n does not necessarily tend to zero, it does not grow exponentially and so the sum defined by (B.4) converges for all $N\geqq 1$. Hence there exists a bounded solution for the particular integral.[26]

The characteristic function is the solution to

$$\left[\frac{r}{\alpha}+\lambda(1-\lambda)^{N-1}\right]V(N)=(1-\lambda)^{2N}[V(N+1)-V(N)]$$
$$+[1-(1-\lambda)^{N-1}][1-(1-\lambda)^N][V(N-1)-V(N)]. \tag{B.5}$$

It is straightforward to show that asymptotically, the characteristic function takes the form

$$V(N)=A_1\left(\frac{1}{1+r/\alpha}\right)^N+A_2(1-\lambda)^{-N^2+N}\left(1+\frac{r}{\alpha}\right)^N, \tag{B.6}$$

where A_1 and A_2 are the constants of integration. Notice we have one stable root and one unstable root. Boundedness of the solution (B.3) requires $A_2=0$. Let ψ_N denote the saddle path solution to (B.5), normalized so that $\psi_N\to 1/(1+r/\alpha)^N$ as $N\to\infty$. Then the solution to (B.1), (B.2) and (B.3) is

$$V(N)=\textstyle\sum_{n=1}^{\infty}a_n(1-\lambda)^{nN}+c\psi_N,$$

where c, the remaining constant of integration, is chosen so that $V(N)$ satisfies (B.2). ||

Lemma B.3. *The unique solution to* (B.1)–(B.3) *is such that for all* $N\geqq 1$, $V(N)\in(0, 1/2)$.

Proof. In order to establish this lemma we need two results, which are stated as Steps 1 and 2.

Step 1. *There does not exist an* $N\geqq 2$ s.t. $V(N)\geqq V(N-1)$ *and* $V(N)\geqq 1/2$.

Proof of Step 1. We argue by contradiction. Suppose there exists an $N^*\geqq 2$ s.t. $V(N^*)\geqq V(N^*-1)$ and $V(N^*)\geqq 1/2$. We rewrite (B.1) as

$$(1-\lambda)^{2N}[V(N+1)-V(N)]=\left(\frac{r}{\alpha}\right)V(N)+\lambda(1-\lambda)^{N-1}[V(N)-\tfrac{1}{2}]$$
$$+[1-(1-\lambda)^{N-1}][1-(1-\lambda)^N][V(N)-V(N-1)]. \tag{B.7}$$

By putting $N=N^*$ into (B.7), it follows that $V(N^*+1)-V(N^*)>0$. Using (B.7) repeatedly, it follows by induction that $V(N+1)>V(N)$ and $V(N)\geqq 1/2$ for *all* $N\geqq N^*$. But (B.3) implies $V(N)$ is bounded. Hence, $V(N)$ must converge to some limit V^*, with $V^*>1/2$. However, as $N\to\infty$, (B.1) implies that if such a limit exists, then $V^*=0$, which is a contradiction. ||

26. Notice that this is not true if there exists $n\geqq 1$ where $1/(1+r/\alpha)=(1-\lambda)^n$. For simplicity, we assume that this problem does not arise. More generally, if it does, the problem is that the particular integral and the characteristic function "coincide" (the characteristic function decays asymptotically at rate $1/(1+r/\alpha)$) and we would have to use a different (and more complicated) guess for the particular integral.

Step 2. *There does not exist an* $N \geqq 2$ s.t. $V(N) \leqq V(N-1)$ *and* $V(N) \leqq 0$.

Proof of Step 2. We argue by contradiction. Suppose there exists an $N^* \geqq 2$ s.t. $V(N^*) \leqq V(N^*-1)$ and $V(N^*) \leqq 0$. Putting $N = N^*$ into (B.7) implies $V(N^*+1) < V(N^*)$. Again by induction (using (B.7)), it follows that $V(N)$ is strictly decreasing, and this time, boundedness implies that $V(N)$ tends to some limit point V^{**}, where $V^{**} < 0$ (strict). (B.1) again implies that such a limit point must be zero, which establishes a contradiction. ||

The proof of the lemma can now be established. It is trivial that $0 < V(1) < 1/2$. [If $V(1) \geqq 1/2$, then (B.2) implies $V(2) > V(1) \geqq 1/2$, which contradicts Step 1. If $V(1) \leqq 0$, then (B.2) implies $V(2) < V(1) \leqq 0$, which contradicts Step 2.] Using Steps 1 and 2, it then follows by induction that $V(N) \in (0, 1/2)$ for all $N \geqq 1$. ||

Proof of Proposition 3.

To prove this proposition, we first need the following result which is a corollary to Lemmas B.2 and B.3.

Corollary C.1. *As* $N \to \infty$, $V(N)$ *declines exponentially, where*

(a) *if* $r/a < \lambda/(1-\lambda)$, $V(N) \to A/(1+r/a)^N$ *for some* $A > 0$, *and*
(b) *if* $r/a > \lambda/(1-\lambda)$, $V(N) \to a_1(1-\lambda)^N$,

where $a_1 > 0$ *is defined in the proof of Lemma B.2.*

Proof of Corollary C.1. By the proof to Lemma B.2, we know that the asymptotic solution for $V(N)$ is $V(N) = a_1(1-\lambda)^N + c/(1+r/a)^N$, where all other terms are of smaller order. Now if $1/(1+r/a) > 1-\lambda$, the second term dominates asymptotically, and Lemma B.3 requires $c > 0$. Otherwise, the first term dominates, but by definition, $a_1 > 0$ in that case. ||

Corollary C.1 implies $V(N)$ is strictly decreasing for N large enough. The proof to the proposition now depends on two steps.

Step 1. $V(N)$ *is decreasing for N large enough and* either $V(N)$ *is strictly decreasing in N for all* $N \geqq 1$, or $V(N)$ *is single-peaked, where* $V(2) \geqq V(1)$.

Proof of Step 1. From Corollary C.1 above it follows that there exists $N^* \geqq 1$ s.t. for all $N \geqq N^*$, $V(N) > V(N+1)$. Define $1 \leqq \bar{N} \leqq N^*$ s.t. for all $N \geqq \bar{N}$, $V(N) > V(N+1)$ and $V(\bar{N}) \geqq V(\bar{N}-1)$. Clearly if $\bar{N} = 1$ then $V(N)$ is strictly decreasing in N for all $N \geqq 1$.

Now suppose $\bar{N} \geqq 2$. To prove $V(N)$ is single peaked, we use backward induction. To do this, rewrite (B.1) as

$$\theta_N[V(N-1) - V(N)] = \left(\frac{r}{a}\right)V(N) - \lambda(1-\lambda)^{N-1}[\tfrac{1}{2} - V(N)] + (1-\lambda)^{2N}[V(N) - V(N+1)], \tag{C.1}$$

where $\theta_N = [1-(1-\lambda)^{N-1}][1-(1-\lambda)^N]$ is a positive number. Using the definition of $\bar{N}$ and putting $N = \bar{N}$ in (C.1) implies

$$\left(\frac{r}{a}\right)V(\bar{N}) - \lambda(1-\lambda)^{\bar{N}-1}[\tfrac{1}{2} - V(\bar{N})] + (1-\lambda)^{2\bar{N}}[V(\bar{N}) - V(\bar{N}+1)] \leqq 0. \tag{C.2}$$

Now consider $N = \bar{N}-1$ in (C.1). Using $V(\bar{N}-1) \leqq V(\bar{N})$ in (C.1) and comparing with (C.2) (noting that $V(\bar{N}) > V(\bar{N}+1)$) it follows that $V(\bar{N}-2) < V(\bar{N}-1)[\leqq V(\bar{N})]$. The induction then follows automatically. For example, put $N = \bar{N}-2$ in (C.1). Using $V(\bar{N}-2) < V(\bar{N}-1) \leqq V(\bar{N})$ and (C.2), it follows that $V(\bar{N}-3) < V(\bar{N}-2)$. Hence $V(N)$ is single-peaked and $V(2) \geqq V(1)$. ||

Step 2. *If* $V(2) \geqq V(1)$, *then* $V(N)$ *is strictly increasing for all N.*

Proof of Step 2. (B.2) implies $V(2) \geqq V(1)$ iff $V(1) \geqq 0.5\lambda/(\lambda + r/a)$. We now use forward induction. Note that (B.7) implies

$$\begin{aligned}(1-\lambda)^{2N}[V(N+1) - V(N)] = {} & (r/a)V(N) - \lambda(1-\lambda)^{N-1}[\tfrac{1}{2} - V(N)] \\ & + [1-(1-\lambda)^{N-1}][1-(1-\lambda)^N][V(N) - V(N-1)].\end{aligned}$$

Using $N=2$ in (B.7) and $V(2) \geqq V(1) \geqq 0{\cdot}5\lambda/(\lambda+r/a)$ implies $V(3)>V(2)$. Now suppose $V(k)>V(k-1)>\cdots>V(2) \geqq V(1) \geqq 0{\cdot}5\lambda/(\lambda+r/a)$ where $k \geqq 3$ and put $N=k$ in (B.7). Then (B.7) implies

$$(1-\lambda)^{2k}[V(k+1)-V(k)] > \left(\frac{r}{a}\right)V(k) - \lambda(1-\lambda)^{k-1}[\tfrac{1}{2}-V(k)]$$

$$> [r/a+\lambda(1-\lambda)^{k-1}]\left[\frac{0{\cdot}5\lambda}{\lambda+r/a}\right] - 0{\cdot}5\lambda(1-\lambda)^{k-1}$$

$$= \frac{0{\cdot}5\lambda(r/a)[1-(1-\lambda)^{k-1}]}{\lambda+r/a} > 0.$$

Hence $V(k+1)>V(k)$ and, by induction, $V(N)$ is increasing for all N. ||

Using Steps 1 and 2, a contradiction proof now establishes the proposition. ||

Proof of Proposition 5.

We prove this proposition by first establishing the following lemma.

Lemma C.1. *For any* $\varepsilon>0$, $V(1, r/a, \cdot) > V(1, r/a+\varepsilon, \cdot)$.

Proof. As in the proof of Lemma B.1, define $G_N(x)$ as the sequence of functions where $G_1(x)=x$ and

$$\left(\frac{r}{a}+\varepsilon\right)G_N(x) = \lambda(1-\lambda)^{N-1}[\tfrac{1}{2}-G_N(x)] + (1-\lambda)^{2N}[G_{N+1}(x)-G_N(x)]$$
$$+[1-(1-\lambda)^{N-1}][1-(1-\lambda)^N][G_{N-1}(x)-G_N(x)], \tag{C.3}$$

subject to the initial condition

$$\left(\frac{r}{a}+\varepsilon\right)x = \lambda[\tfrac{1}{2}-x] + (1-\lambda)^2[G_2(x)-x], \tag{C.4}$$

(here we have replaced r/a with $r/a+\varepsilon$).

(i) First consider $x=V(1, r/a, \cdot)$. Comparing (C.4) to (B.2)

$$G_2(x) - V(2, r/a, \cdot) = \frac{\varepsilon V(1, r/a, \cdot)}{(1-\lambda)^2} > 0. \tag{C.5}$$

Moreover, using (C.5) and $G_1(x)=x=V(1, r/a, \cdot)$, it can be shown by induction on (C.3) and (B.1) that

$$G_{N+1}(x) - G_N(x) > V(N+1, r/a, \cdot) - V(N, r/a, \cdot) \qquad \text{for all } N \geqq 2.$$

Repeated summation of this inequality (over N) implies that

$$G_{N+1}(x) - V(N+1, r/a, \cdot) > G_2(x) - V(2, r/a, \cdot) > 0.$$

This, in turn, implies that, for all $N \geqq 2$, $G_{N+1}(x) > G_2(x) - V(2, r/a, \cdot) > 0$.

(ii) Now consider $x=V(1, r/a+\varepsilon, \cdot)$. Then, $G_N(x) \equiv V(N, r/a+\varepsilon, \cdot)$ and hence $\lim_{N\to\infty} G_N(x)=0$. The proof to Lemma B.1 establishes that $dG_N/dx>0$ for all x and N. Hence (i) and (ii) imply the lemma. ||

We can now prove the proposition by contradiction. Suppose it is not true. Then Lemma C.1 implies there exists an $N^* \geqq 2$ s.t. $V(N^*-1, r/a, \cdot) > V(N^*-1, r/a+\varepsilon, \cdot)$ *and* $V(N^*, r/a+\varepsilon, \cdot) \geqq V(N^*, r/a, \cdot)$. Again consider (C.3) and set $x=V(1, r/a+\varepsilon, \cdot)$ so that $G_N(x) \equiv V(N, r/a+\varepsilon, \cdot)$. Using (C.3) and (B.1), the above inequalities imply

$$G_{N^*+1}(x) - G_{N^*}(x) > V(N^*+1, r/a, \cdot) - V(N, r/a, \cdot) \text{ (strict)}.$$

Again it follows by induction on (C.3) and (B.1) that

$$G_{N+1}(x) - G_N(x) > V(N+1, r/a, \cdot) - V(N, r/a, \cdot) \qquad \text{for all } N \geqq N^*.$$

As in the proof to Lemma C.1, repeatedly sum these inequalities over N. Then, taking the limit as $N \to \infty$, it follows that: $\lim_{N\to\infty} G_N(x) \equiv \lim_{N\to\infty} V(N, r/\alpha+\varepsilon, \cdot) > V(N^*+1, r/\alpha+\varepsilon, \cdot) - V(N^*+1, r/\alpha, \cdot)$, which is strictly positive. But this contradicts $\lim_{N\to\infty} V(N, r/\alpha+\varepsilon, \cdot) = 0$. ||

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